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New example of a generating function for WZNW fusion rules

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Abstract. We review the construction of generating functions for WZNW (Wess–Zumino–Novikov–Witten) fusion rules and present the explicit form of the generating function for $\widehat{so}(5)$. In addition to providing a compact and manifestly positive expression for the $\widehat{so}(5)$ fusion rules at all levels, this result illustrates a general conjecture on the relation between the generating function for WZNW fusion rules and that for finite Lie algebra tensor products.

1. Introduction

In a rational conformal field theory, the fusion coefficient counts the number of independent couplings between three given primary fields. It can also be thought of as arising from a product $\times$ defined as

$$\lambda \times \mu = \bigoplus_{\nu} N_{\lambda\mu}{}^{\nu}\nu \tag{1.1}$$

where λ, μ and ν denote primary fields. $N_{\lambda\mu}{}^{\nu}$ is a fusion coefficient and (1.1) is known as a fusion rule.

The building blocks of a large class of unitary rational conformal field theories are the Wess–Zumino–Novikov–Witten (WZNW) models. The current algebra of these models is an affine Kac–Moody algebra $\widehat{g}$ at integer level k. The WZNW primary fields are in one-to-one correspondence with integrable representations of $\widehat{g}$ at level k [1, 2] (so that both may be specified by the same symbol). Over the past years various methods have been proposed for the calculation of WZNW fusion rules. These are reviewed briefly later, after we have introduced some notation.

Notation. Let g denote the horizontal finite Lie subalgebra of $\widehat{g}$. If ω^{μ} ($\mu = 0, 1, \ldots, r = \mathrm{rank}(g)$) are the fundamental weights of $\widehat{g}$, then an affine weight may be written as $\lambda = \sum_{\mu=0}^{r} \lambda_{\mu}\omega^{\mu} =: [\lambda_0, \lambda_1, \ldots, \lambda_r]$. We can associate with such a weight λ a weight $\bar{\lambda}$ of the finite algebra g: $\bar{\lambda} = \sum_{i=1}^{r} \lambda_i \bar{\omega}^i =: (\lambda_1, \ldots, \lambda_r)$, where $\bar{\omega}^i (i = 1, \ldots, r)$ are the fundamental weights of g. We call $\bar{\lambda}$ the finite part of λ. The level k of the weight λ is $k = \sum_{\mu=0}^{r} \lambda_{\mu} k^{\vee\mu}$ where the $k^{\vee\mu}$ are the co-marks ($k^{\vee 0} := 1$, and the remaining $k^{\vee i}$ are the coefficients of expansion of the longest root of g—denoted by $\bar{\theta}$—in terms of the simple co-roots). So, given $\bar{\lambda}$ we can always reconstruct λ for a fixed level k. We also call λ the affine extension of $\bar{\lambda}$.

If an affine weight λ has Dynkin labels obeying $\lambda_\mu \in \mathbf{Z}_+$ and is of level k, it is the highest weight of an integrable $\widehat{g}$ representation at level k. The set of such weights is denoted P_+^k. The highest weights of integrable representations of g have Dynkin labels $\lambda_i \in \mathbf{Z}_+ (i = 1, \ldots, r)$. The set of all such integrable weights of g is denoted P_+. Note that if $\lambda \in P_+^k$ then $\bar{\lambda} \in P_+$, but the converse is not generally true. $\bar{\lambda} \in P_+$ does not imply $\lambda \in P_+^k$, since λ may have negative λ_0 if k is sufficiently small. Another way of saying this last point is $\bar{P}_+^k \subset P_+$.

Finally, we denote by $C\nu$ the highest weight of the representation conjugate to that of ν. Equivalently it indicates the primary field that is the charge conjugate of the primary field ν. In the following we use the relation $N_{\lambda\mu}{}^{\nu} = N_{\lambda\mu C\nu}$.

1.1. The depth rule of Gepner and Witten

Let us denote by $\bar{N}_{\bar{\lambda}\bar{\mu}}{}^{\bar{\nu}}$ the tensor product coefficients associated with the product $\bar{\lambda} \otimes \bar{\mu}$, where $\bar{\lambda}$ is the finite part of λ, etc. Then the depth rule [2] is the statement that $N_{\lambda\mu\nu} = 0$ if one of the following two conditions is satisfied:

$$\bar{N}_{\bar{\lambda}\bar{\mu}\bar{\nu}} = 0 \tag{1.1a}$$

$$d_{\lambda'} + d_{\mu'} + d_{\nu'} > k \qquad \text{where} \qquad \lambda' + \mu' + \nu' = 0. \tag{1.1b}$$

Here λ' is a weight in the representation whose highest weight is λ (similarly for μ' and ν'). $d_{\lambda'}$ is its depth. Strictly speaking, the depth of a weight does not make sense. However, to each λ' one can associate a number of states $|\lambda'(i)\rangle$ in the g representation λ, the number being equal to the multiplicity of λ'. The depth of a state $|\lambda'(i)\rangle$ is simply the maximum number $d_{\lambda'(i)}$ such that $(J_{-\alpha_0})^{d_{\lambda'(i)}}|\lambda'(i)\rangle \neq 0$, where $J_{-\alpha_0}$ is the lowering operator corresponding to the zeroth simple root α_0.

Now in order for λ', μ', ν' to be a triplet of weights relevant to the coupling $\lambda \times \mu \times \nu$ it is necessary that $\bar{\lambda}' + \bar{\mu}' + \bar{\nu}' = 0$ since we want $\bar{\lambda} \otimes \bar{\mu} \otimes \bar{\nu} = 0 \oplus \ldots$, but it is not sufficient. We must furthermore have a coupling between $|\lambda'(m)\rangle, |\mu'(n)\rangle, |\nu'(p)\rangle$. If one can associate a coupling to this triplet of states, then the corresponding WZNW coupling will not exist if $k < d_{\lambda'(m)} + d_{\mu'(n)} + d_{\nu'(p)}$.

The practical implementation of this rule is intricate. First, the depths themselves are difficult to calculate. Furthermore, it is hard to specify the triplets of states $|\lambda'(m)\rangle|\mu'(n)\rangle|\nu'(p)\rangle$ that correspond to a given g-coupling. Up to now the depth rule has been used to calculate fusion rules for $\widehat{\mathrm{su}}(2)$ [2], and some examples (at low levels) in $\widehat{\mathrm{su}}(3)$ [2] and $\widehat{\mathrm{E}}_8$ [3].

Further discussion of this rule has been presented by Gepner in [4] and will be reported in [5].

1.2. The Verlinde formula

Verlinde [6] discovered that the fusion rules are diagonalized by the modular transformation $S : \tau \rightarrow -1/\tau$. If $S_{\lambda\xi}$ describes how characters transform, one consequence is the following formula for the fusion rules

$$N_{\lambda\mu\nu} = \sum_{\xi \in P_+^k} \frac{S_{\lambda\xi} S_{\mu\xi} S_{\nu\xi}}{S_{0\xi}} \tag{1.2}$$

where the subscript 0 indicates the identity primary field (i.e. $k\omega^0$). From this formula, it is not obvious that $N_{\lambda\mu\nu} \in \mathbf{Z}_+$ since the matrix elements of S may be complex numbers with irrational real and imaginary parts†.

1.3. Using the affine Weyl group

The Weyl group of g may be used to calculate in an efficient way the tensor product coefficient $\bar{N}$ [7]. To calculate the fusion coefficients, the Weyl group of $\widehat{g}$ (the affine Weyl group) may similarly be used. The algorithm proposed in [8] (see also [9]) leads to the following relation between the fusion rules and the tensor product coefficients [10–12]:

$$N_{\lambda\mu}{}^{\nu} = \sum_{\substack{\xi \in P_+ \\ w.\xi = \nu}} \bar{N}_{\bar{\lambda}\bar{\mu}}{}^{\bar{\xi}} \epsilon(w). \tag{1.3}$$

w is an element of the affine Weyl group, of sign $\epsilon(w)$, and the dot indicates the shifted action, i.e. $w.\lambda = w(\lambda + \rho) - \rho$ where $\rho = \sum_{\mu=0}^{r} \omega^{\mu}$. This seems to be the simplest way of calculating WZNW fusion rules. A drawback of this formula is that it is not manifestly positive. It does, however, make clear that $N_{\lambda\mu\nu} \in \mathbf{Z}$.

1.4. Generating functions

A generating function for fusion rules is a sum over all possible couplings whose expansion in power series allows us to read off the coefficients $N_{\lambda\mu\nu}$. It may be written in a manifestly positive form, showing clearly that $N_{\lambda\mu\nu} \in \mathbf{Z}_+$. Generating functions for WZNW fusion rules were introduced in [13] and explicit examples were given for $\widehat{\mathrm{su}}(2)$ and $\widehat{\mathrm{su}}(3)$. They provide an elegant and compact presentation of the fusion rules. However, this approach also has its drawbacks. First, one needs to know the corresponding generating function of the tensor product coefficients. But these are known only for the lowest rank Lie algebras, since their complexity increases very quickly with the rank. Second, the generating function for fusion rules is related to one‡ of the possible manifestly positive expressions of the generating function for tensor product coefficients. But there is no way *a priori* of selecting the required manifestly positive form. (This will be discussed in more detail in the next section.)

For completeness we should say that WZNW fusion rules can also be calculated from Wakimoto type free field representations [14]. Also, for su(3) manifestly positive combinatorial methods have been found by Cummins [15] and Lu [16].

In this work we present further results on the fourth approach. More precisely, we provide a new example of a generating function for WZNW fusion rules, that for $\widehat{g} = \widehat{\mathrm{so}}(5)$, in a manifestly positive form. It is displayed in section 3. Although we do

† Indeed, up to a constant (fixed by unitarity) the modular matrix S for $\widehat{g}$ reads [11, 21]

$$S_{\lambda\mu} \sim \sum_{w \in W(g)} \epsilon(w) \exp\left[\frac{-2\pi i}{(m + g_c)} (w(\lambda + \rho), \mu + \rho)\right]$$

where g_c is the dual Coxeter number, $\epsilon(w)$ is the sign of the Weyl group element w and $W(g)$ is the finite Weyl group.

‡ More precisely, it could be related to more than one although one is enough.

not have a formal proof of its validity, it has been tested to a sufficiently high order ($k \leqslant 11$) to make us confident it is correct. We tested it by comparing it with the results obtained from formula (1.3). For completeness, in an appendix, we illustrate the latter method, including the calculation of $\bar{N}$ by Young tableaux. Our new example of a WZNW fusion rule generating function provides a non-trivial test of the general conjectures formulated in [13], reviewed in section 3. Section 2 contains the necessary information on generating functions for tensor product coefficients.

2. Generating functions for tensor product coefficients

The generating function for tensor product coefficients is given by a sum over all possible couplings [17]

$$\bar{G}(L,M,N) := \sum_{\bar{\lambda},\bar{\mu},\bar{\nu}\in P_+} \bar{N}_{\bar{\lambda}\bar{\mu}\bar{\nu}} L^{\bar{\lambda}} M^{\bar{\mu}} N^{\bar{\nu}} \tag{2.1}$$

where

$$L^{\bar{\lambda}} := \prod_{i=1}^{r} L_i^{\lambda_i} \tag{2.2}$$

and similarly for $M^{\bar{\mu}}$ and $N^{\bar{\nu}}$.

For example the generating function for $g = \mathrm{su}(2)$ ($r = 1$) reads [17]

$$\bar{G} = [(1 - L_1M_1)(1 - L_1N_1)(1 - M_1N_1)]^{-1}. \tag{2.3}$$

To read off coefficients one expands all the terms in a power series and compares them with (2.1). More explicitly, suppose that we want to find the tensor product $(\lambda_1)\otimes(\mu_1)$. Then in the series expansion we collect all the terms of the form $L_1^{\lambda_1} M_1^{\mu_1} N_1^{\nu_1}$. All the values of ν_1 which we encounter in this way are those which appear in the tensor product of (λ_1) with (μ_1). For example it is very easy to see that the only terms containing exactly one factor of L_1 and one factor of M_1 are L_1M_1 and $L_1M_1N_1^2$ which means that $(1) \otimes (1) = (0) \oplus (2)$. The simplicity of the generating function follows from the fact that any coupling can be written as a product of elementary couplings. For su(2) these elementary couplings are

$$E_1 = L_1M_1 \qquad E_2 = L_1N_1 \qquad E_3 = M_1N_1. \tag{2.4}$$

Let us consider another example. The generating function of $g = \mathrm{su}(3)$ ($r = 2$) is [17]

$$\begin{aligned} \bar{G} = {} & [(1 - L_1N_2)(1 - L_2M_1)(1 - M_2N_1)(1 - L_1M_1N_1)(1 - L_2M_2N_2)]^{-1} \\ & \times \left(1 + \frac{L_1M_2}{(1 - L_1M_2)(1 - L_2N_1)} + \frac{M_1N_2}{(1 - M_1N_2)(1 - L_1M_2)} \right. \\ & \left. + \frac{L_2N_1}{(1 - L_2N_1)(1 - M_1N_2)}\right). \end{aligned} \tag{2.5}$$

Notice that this is a manifestly positive expression. For su(3) there are eight elementary couplings:

$$\begin{array}{lll} E_1 = L_1M_2 & E_2 = L_1N_2 & E_3 = M_1N_2 \\ E_4 = L_2M_1 & E_5 = L_2N_1 & E_6 = M_2N_1 \\ E_7 = L_1M_1N_1 & E_8 = L_2M_2N_2. & \end{array} \tag{2.6}$$

In contrast to the su(2) case, this generating function cannot be written in the form $\prod_i(1-E_i)^{-1}$. This is because the expression for a given coupling as a product of elementary couplings is in general not unique. These redundancies are called syzygies. They can be avoided by forbidding certain couplings, but this choice is not unique. Different sets of forbidden couplings yield different manifestly positive forms of $\bar{G}$. Of course these different forms must all be equivalent: $\bar{G}$ as a function of the variables L_i, M_i and N_i is unique! For example the expression (2.5) is obtained by forbidding the coupling $E_1E_3E_5$. To see this explicitly, let us rewrite (2.5) in terms of the elementary couplings:

$$\bar{G} = \left(\prod_{i\neq 1,3,5}(1-E_i)^{-1}\right)\left(1+\frac{E_1}{(1-E_1)(1-E_5)}+\frac{E_3}{(1-E_3)(1-E_1)} + \frac{E_5}{(1-E_5)(1-E_3)}\right). \tag{2.7}$$

It is clear that in expanding this form, we will never encounter a term containing a product of the three factors $E_1E_3E_5$. (The factors in the numerators are necessary to avoid overcounting.) Now since, for instance $E_1E_3E_5 = E_7E_8$, we could equivalently forbid the coupling E_7E_8, yielding

$$\bar{G} = \left(\prod_{i\neq 7,8}(1-E_i)^{-1}\right)\left(1+\frac{E_7}{(1-E_7)}+\frac{E_8}{(1-E_8)}\right). \tag{2.8}$$

Then clearly no factors containing E_7E_8 can appear. Therefore, although $\bar{G}(L,M,N)$ is unique, its expression in terms of the E_is is not, because the syzygies may be used in different ways.

As a final example we present the generating function for $g = \text{so}(5) = \text{sp}(4)$ $(r=2)$ [18]. We choose the first root to be the short one. The expression for $\bar{G}$ given in [18] is

$$\begin{aligned} \bar{G} = {}& [(1-M_1N_1)(1-L_1N_1)(1-L_1M_1)(1-M_2N_2)(1-L_2N_2)(1-L_2M_2)]^{-1} \\ & \times\left(\frac{1}{(1-L_2M_1N_1)(1-L_2M_1^2N_2)}+\frac{L_2M_2N_1^2}{(1-L_2M_1N_1)(1-L_2M_2N_1^2)}\right. \\ & +\frac{L_1^3M_2^2N_1N_2}{(1-L_1M_2N_1)(1-L_1^2M_2N_2)}+\frac{L_1M_2N_1}{(1-L_1M_2N_1)(1-L_2M_2N_1^2)} \\ & \left.+\frac{L_1^2M_2N_2}{(1-L_1M_1N_2)(1-L_1^2M_2N_2)}+\frac{L_1M_1N_2}{(1-L_1M_1N_2)(1-L_2M_1^2N_2)}\right). \end{aligned} \tag{2.9}$$

A convenient basis for the elementary couplings is

$$\begin{aligned} &A_1 = M_1N_1 \qquad A_2 = L_1N_1 \qquad A_3 = L_1M_1 \\ &B_1 = M_2N_2 \qquad B_2 = L_2N_2 \qquad B_3 = L_2M_2 \\ &C_1 = L_2M_1N_1 \qquad C_2 = L_1M_2N_1 \qquad C_3 = L_1M_1N_2 \\ &D_1 = L_1^2M_2N_2 \qquad D_2 = L_2M_1^2N_2 \qquad D_3 = L_2M_2N_1^2. \end{aligned} \tag{2.10}$$

This expression for $\bar{G}$ follows by forbidding the following set of couplings

$$\{C_iC_j, D_iD_j, C_iD_i\} \qquad i,j = 1,2,3 \text{ and } i \neq j. \tag{2.11}$$

Thus nine couplings must be forbidden. But again this set is far from unique. In the following we will use the compact notation

$$\tilde{E}_i := (1 - E_i)^{-1} \tag{2.12}$$

in terms of which (2.9) reads

$$\bar{G} = \left(\prod_{i=1}^{3} \tilde{A}_i\tilde{B}_i\right)(\tilde{C}_1\tilde{D}_2 + D_3\tilde{C}_1\tilde{D}_3 + C_2D_1\tilde{C}_2\tilde{D}_1 + C_2\tilde{C}_2\tilde{D}_3 + D_1\tilde{C}_3\tilde{D}_1 + C_3\tilde{C}_3\tilde{D}_2). \tag{2.13}$$

3. Generating functions for fusion rules

We define fusion rule generating functions as follows [13]

$$G(L,M,N;d) = \sum_{k=0}^{\infty} d^k \sum_{\lambda,\mu,\nu \in P_+^k} N_{\lambda\mu\nu} L^{\bar{\lambda}} M^{\bar{\mu}} N^{\bar{\nu}} \tag{3.1}$$

where the extra dummy variable d keeps track of the level.

Let us now review how this generating function can be constructed from $\bar{G}$, according to the general conjectures presented in [13]. The first conjecture is that each three-point coupling in a WZNW model exists for levels $k \geqslant k_0$ where k_0 is a certain minimum level. Now let e_i be the minimum level for the elementary coupling E_i. The second conjecture in [13] is that there always exists a set of forbidden couplings for which G and $\bar{G}$ are related in a simple way. Let F be this appropriate set of forbidden couplings and denote by $\bar{G}_F(E_i)$ the corresponding manifestly positive expression for the tensor product coefficients. Then the conjectured relation between G and $\bar{G}$ is

$$G(L,M,N;d) = (1-d)^{-1}\bar{G}_F(d^{e_i}E_i) \tag{3.2}$$

The factor $(1-d)^{-1}$ takes care of the elementary coupling of three scalars which is present at all levels. This provides a manifestly positive and compact expression for the fusion rules.

For su(2) there are no forbidden couplings and the elementary couplings all have minimum level equal to 1, which yields [13]:

$$G(L,M,N;d) = (1-d)^{-1}\prod_{i=1}^{3}(\widetilde{dE_i}) = [(1-d)(1-dL_1M_1)(1-dL_1N_1)(1-dM_1N_1)]^{-1}. \tag{3.3}$$

It is easy to show that this reproduces the correct $\widehat{su}(2)$ fusion rules at all levels [2, 19]. The su(2) fusion rules coincide with the tensor products, except for the 'truncation from above'; the fusion coefficient vanishes unless $2k \geqslant \lambda_1 + \mu_1 + \nu_1$. This result leads directly to (3.3).

For su(3), $e_i = 1$ again for all elementary couplings, but now there are essentially (i.e. up to charge conjugation) two possibilities for the forbidden couplings. These lead to two different manifestly positive forms for $\bar{G}$: (2.7) and (2.8). These two forms can be written compactly in a non-manifestly positive way as

$$\bar{G} = \left(\prod_{i=1}^{8}\tilde{E}_i\right)(1-F) \tag{3.4}$$

where F is the forbidden coupling, that is either $E_1E_3E_5$ or E_7E_8. Now if we apply the prescription (3.2) we obtain

$$G = \tilde{d}\prod_{i=1}^{8}(\widetilde{dE_i})(1-d^aF) \tag{3.5}$$

where

$$\begin{aligned} a &= 2 \qquad \text{if} \qquad F = E_7E_8 \\ a &= 3 \qquad \text{if} \qquad F = E_1E_3E_5. \end{aligned} \tag{3.6}$$

Therefore in order to completely fix G, we only have to analyse the coupling $L_1L_2M_1M_2N_1N_2$ to see which of the two possible values of a yield the correct results. It is simple to verify that

$$(1,1)\otimes(1,1) = (0,0)_2 \oplus (1,1)_2 \oplus (1,1)_3 \oplus (3,0)_3 \oplus (0,3)_3 \oplus (2,2)_4 \tag{3.7}$$

where the subscript indicates the value of k_0, the minimum level at which the three point coupling is allowed. Thus one coupling $L_1L_2M_1M_2N_1N_2$ arises at level 2 and another appears at level 3. Now the piece $\tilde{d}\prod_i(\widetilde{dE_i})$ already yields one such coupling at level 2 but two at level 3. Thus one factor $d^3L_1L_2M_1M_2N_1N_2$ must be cancelled, which fixes the value a to be 3 so that the correct choice of F is $E_1E_3E_5$ [13]. That this indeed gives the correct form of the generating function for $\widehat{su}(3)$ has been established rigorously in [15].

Let us now turn to the main subject of this work, the derivation of the WZNW generating function for so(5). The first natural test is to apply the prescription (3.2) to the form (2.13). It is easy to verify that elementary couplings A_i, B_i and C_i have minimum level one while the D_is have minimum level 2. Thus we just multiply all elementary couplings by the appropriate power of d and multiply the result by

$(1-d)^{-1}$. This is then a candidate for G. However if we look at the coupling $(1,1)(1,1)(2,0)$ one can see readily that this is not the correct choice. Indeed one has (see appendix)

$$(1,1)\otimes(1,1)=(0,0)_2\oplus(0,1)_2\oplus(2,0)_2\oplus(0,2)_3\oplus(2,0)_3\oplus(0,3)_3\oplus 2(2,1)_3 \oplus(2,2)_4\oplus(4,0)_4. \tag{3.8}$$

Hence (2,0) must appear at level 2 in the product $(1,1)\otimes(1,1)$ which means that a term $d^2L_1L_2M_1M_2N_1^2$ must appear in the power series expansion of G. However the G constructed from (2.13)—which corresponds to the set of forbidden couplings (2.11)—does not contain such a term. The lowest power of d multiplying $L_1L_2M_1M_2N_1^2$ that it produces is 3, and it comes from the coupling A_3D_3.

The only way of obtaining the coupling $L_1L_2M_1M_2N_1^2$ at level 2 would be from the product C_1C_2 (which would then come with a factor d^2). But with the choice (2.11), this is forbidden. Therefore one must look for an other set of forbidden couplings which would not include C_iC_j. Reference [18] gives us the syzygies. The relation important to us is that C_iC_j is a linear combination of A_kD_k and $A_iA_jB_k$ (here ijk are 123 in any order). So, instead of (2.11) we consider the set

$$\{A_kD_k, D_iD_j, C_iD_i\}. \tag{3.9}$$

We found that the corresponding $\tilde{G}$ could be written in essentially two equivalent ways:

$$\tilde{G}=\left(\prod_{i=1}^{3}\tilde{B}_i\right)\left(\left(\prod_{i=1}^{3}\tilde{A}_i\tilde{C}_i\right)(1-S)+D_1\tilde{D}_1\tilde{A}_2\tilde{A}_3\tilde{C}_2\tilde{C}_3+D_2\tilde{D}_2\tilde{A}_1\tilde{A}_3\tilde{C}_1\tilde{C}_3 + D_3\tilde{D}_3\tilde{A}_1\tilde{A}_2\tilde{C}_1\tilde{C}_2\right) \tag{3.10}$$

with

$$S=C_1C_2C_3 \qquad \text{or} \qquad A_2A_3B_1C_1 \tag{3.11}$$

(where the second case could appear with the indices permuted). When written in terms of the L_i, M_i and N_i, it gives the same result as (2.13). With the replacement† $E_i\to d^{e_i}E_i$, $C_1C_2C_3$ will appear with a factor d^3 while $A_2A_3B_1C_1$ will have a d^4. Thus one only has to go to level 3 and look at the coupling $L_1^2L_2M_1^2M_2N_1^2N_2=(2,1)(2,1)(2,1)$ in order to decide which is the correct choice. (Of course, before we checked this, we checked that everything comes out right at level 2). The tensor product $(2,1)\otimes(2,1)$ gives, with the values of k_0 inserted,

$$(2,1)\otimes(2,1)=(0,0)_3\oplus(0,1)_3\oplus(0,2)_3\oplus(2,0)_3\oplus(2,1)_3\oplus 2(2,1)_4\oplus(0,2)_4\oplus(0,3)_4 \oplus(0,4)_4\oplus(2,0)_4\oplus 2(2,2)_4\oplus(4,0)_4\oplus(2,2)_5\oplus(2,3)_5\oplus(4,0)_5\oplus 2(4,1)_5 \oplus(4,2)_6\oplus(6,0)_6. \tag{3.12}$$

† Here we are using the fact that for both choices of S, $\tilde{G}$ may be rewritten as a manifestly positive expression in terms of the E_i.

Hence one coupling $(2,1)(2,1)(2,1)$ appears at level 3 and two more at level 4. Clearly in order to have one such coupling at level 3 we should not choose $S = C_1C_2C_3$ since that would cancel the contribution $d^3C_1C_2C_3$ coming from $\prod_i(\widetilde{dC_i})$. Thus one must take $S = A_2A_3B_1C_1$ (or one of the other two equivalent expressions obtained by permuting indices). We then checked that all the other couplings at level 4 are correctly described by this function.

Thus we have a candidate for G, constructed from the general prescription (3.2) with a suitable choice of forbidden couplings, which works at least up to level 4. It can be written in a manifestly positive form as:

$$G = (1-d)^{-1}\bar{G}(d^{e_i}E_i) \qquad e_i = \begin{cases} 1 & \text{for } A_i, B_i, C_i \\ 2 & \text{for } D_i \end{cases}$$

$$\begin{aligned}\bar{G} = \tilde{B}_2\tilde{B}_3(&\tilde{A}_1\tilde{A}_3\tilde{B}_1\tilde{C}_1\tilde{C}_2\tilde{C}_3 + A_2\tilde{A}_1\tilde{A}_2\tilde{B}_1\tilde{C}_1\tilde{C}_2\tilde{C}_3 + A_2A_3\tilde{A}_1\tilde{A}_2\tilde{A}_3\tilde{C}_1\tilde{C}_2\tilde{C}_3 \\ &+ A_2A_3B_1\tilde{A}_1\tilde{A}_2\tilde{A}_3\tilde{B}_1\tilde{C}_2\tilde{C}_3 + D_1\tilde{A}_2\tilde{A}_3\tilde{B}_1\tilde{C}_2\tilde{C}_3\tilde{D}_1 + D_2\tilde{A}_1\tilde{A}_3\tilde{B}_1\tilde{C}_1\tilde{C}_3\tilde{D}_2 \\ &+ D_3\tilde{A}_1\tilde{A}_2\tilde{B}_1\tilde{C}_1\tilde{C}_2\tilde{D}_3). \end{aligned} \tag{3.13}$$

We have tested this generating function up to level 11 by comparing (with the aid of a computer) the results with those predicted by formula (1.3). We found perfect agreement. Let us now argue that testing the function for level $0 \leqslant k \leqslant 11$ is sufficient to demonstrate its validity.

In order to find the generating function G we could have proceeded by brute force along the following lines. We can write $\bar{G}(L, M, N)$ in a non-manifestly positive form by expressing all the terms with a common denominator, as

$$\bar{G} = \left\{\prod_{i=1}^{3}\tilde{A}_i\tilde{B}_i\tilde{C}_i\tilde{D}_i\right\}\{1 + \cdots + L_1^4L_2^3M_1^4M_2^3N_1^4N_2^3\}. \tag{3.14}$$

Here $\cdots$ stands for 32 terms of the form $L_1^{\lambda_1}L_2^{\lambda_2}M_1^{\mu_1}M_2^{\mu_2}N_1^{\nu_1}N_2^{\nu_2}$ and the term written explicitly is the one with maximal value of $\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \nu_1 + \nu_2$. A form such as this one is possible for any $\bar{G}$, and is unique given a set of elementary couplings. To obtain $(1-d)G$ we could multiply all the terms in (3.14) by a factor d raised to some power and fix the powers by looking at the couplings level by level. This is clearly a finite (though quite lengthy) process, which would end when the power of the term with maximal value of $\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \nu_1 + \nu_2$ was determined. An upper bound for the power of d multiplying a term $L_1^{\lambda_1}L_2^{\lambda_2}M_1^{\mu_1}M_2^{\mu_2}N_1^{\nu_1}N_2^{\nu_2}$ is simply $\frac{1}{2}(\lambda_1+\lambda_2+\mu_1+\mu_2+\nu_1+\nu_2)$ (since this upper bound holds for all elementary couplings). For the last term in (3.14), this is $21/2$. So by using this procedure up to level 11 we fix uniquely the form of G†.

So (3.13) should be the correct generating function for $\widehat{so}(5)$ WZNW fusion rules. The procedure we used to construct it provides a non-trivial test of the main result of [13]: the conjecture that there is a simple relation between G and $\bar{G}$ of the form (3.2), for some set F of forbidden couplings.

† When applied to su(3), this argument shows that we only have to go up to level 3 to fully fix the form of G.

4. Conclusion

We have reviewed the construction of generating functions for fusion rules in WZNW models, first introduced in [13]. By displaying the generating function for $\widehat{so}(5)$, we have demonstrated their existence for Lie algebras more complicated than su(2) and su(3) and that, moreover, this is not a particularity of su(N) Lie algebras. Furthermore we succeeded in constructing this new generating function according to the prescription proposed in [13]. This thus provides further support for the validity of this general scheme.

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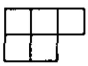
Appendix

A1. Calculation of tensor product coefficients for so(5) using Young tableaux

Since so(5) = sp(4), one can calculate tensor product coefficients for so(5) with Young tableaux techniques for sp($2m$). These are just slightly more complicated than the Littlewood–Richardson rules for the multiplication of Young tableaux for su(N) [20]. Let $\bar{\lambda} = (\lambda_1, \ldots, \lambda_m)$ *denote a sp($2m$) highest weight.* The corresponding Young tableau has λ_1 columns with one box, λ_2 columns with two boxes, etc. To state the multiplication rules we need a representation of the weight in terms of the numbers $\langle m_1, m_2, \ldots \rangle =: \langle m \rangle$ where m_1 is the number of boxes in the first row of the Young tableau, m_2 is the number of boxes in the second row, etc. Thus for example the sp(4) weight $\lambda = (1, 2)$ has the following representation

$(1,2) =$  $= \langle 3, 2 \rangle$

The tensor product is given by

$$\langle m \rangle \otimes \langle n \rangle = \sum_{\langle p \rangle} \langle (m/p) \bullet (n/p) \rangle \tag{A1}$$

where $\bullet$ stands for the Littlewood–Richardson product and / is its inverse. In the summation $\langle p \rangle$ can take the following values

$$\langle p \rangle = \langle 0, 0, 0, \ldots \rangle, \ldots, \langle p_1, p_2, \ldots \rangle \qquad \text{where } p_i = \min(m_i, n_i). \tag{A2}$$

Let us recall the rules for the Littlewood–Richardson product of two Young tableaux [20]. In the second Young tableau we put numbers 1 in all the boxes of the first row, 2 in all boxes of the second row, etc. Then we add all the boxes 1 to the first tableau and keep only the resulting tableaux which satisfy the following two conditions

(i) The resulting tableau must be regular: the number of boxes in a given row must be smaller or equal to the number of boxes in the row just above.

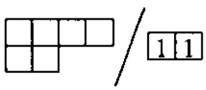
(ii) The resulting tableaux must not contain two boxes marked by 1 in the same column.

The tableaux which do not satisfy these conditions are ignored. To the resulting tableaux, one then adds all boxes marked by a 2 and again one keeps only the tableaux which satisfy (i) and (ii), where now in (ii) 1 is replaced by 2. One continues in this way until all the boxes of the second tableau in the original product have been used. In this process an additional rule must be respected:

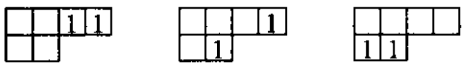
(iii) In counting from right to left and top to bottom, the number of 1s must always be greater than or equal to the number of 2s, the number of 2s must always be greater or equal to the number of 3s, etc.

Finally the resulting tableaux which contain columns with more boxes than the rank of the group (i.e. non-standard tableaux) must be properly modified. Let us describe this modification for sp(4) (and from now on we restrict ourself to this case). Let q_i be the number of boxes in the ith column. Then for sp(4) the modification amounts to taking out of the first column h_1 boxes where $h_1 = 2q_1 - 6$ (recall that for sp(4) $q_i > 2$ for a non-standard tableau) and repeat the procedure for all columns with $q_i > 2$. The modified tableau is $(-)^c\langle m - h\rangle = (-)^c\langle m_1 - h_1, m_2 - h_2\rangle$ where c is the number of columns which have been affected. If after this operation the tableau is still non-standard, it is ignored. For instance a tableau with three boxes all in the first column has $q_1 = 3$ so $h_1 = 0$. Since the tableau is unaffected by the modification (i.e. it remains non-standard), it is rejected. On the other hand a tableau with four boxes all in the first column ($q_1 = 4, h_1 = 2$) is modified into a tableau with two boxes in the first column and it comes with a minus sign.

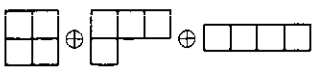
Let us now illustrate the operation / by an example:

$\langle 4,2\rangle/\langle 2,0\rangle =$ 

In the first step we mark two boxes at the edges of the first tableau with 1s. This will tell us how the first tableau can be decomposed into a $\bullet$ product of a smaller tableau times $\langle 2,0\rangle$ so that the division will give as a result this smaller tableau. Of course the marking must respect the rules stated above. Here there are three possibilities:



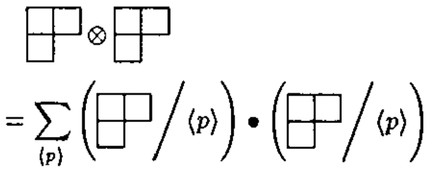
The division then amounts to taking out the two boxes marked by a 1 and the result is

 $= \langle 2,2\rangle + \langle 3,1\rangle + \langle 4,0\rangle$

We are now in position to present an example of the application of the general formulae

(A1) and (A2) for $\bar{g} = \mathrm{sp}(4)$. We will consider the tensor product:

$$(1,1) \otimes (1,1) = \langle 2,1 \rangle \otimes \langle 2,1 \rangle$$

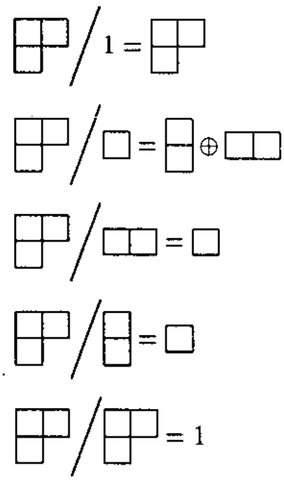


In the summation, $\langle p \rangle$ can take the following values

$\langle p \rangle = \langle 0,0 \rangle \quad \langle 1,0 \rangle \quad \langle 1,1 \rangle \quad \langle 2,0 \rangle \quad \langle 2,1 \rangle$

$= 1$

Thus we need the following divisions:



We then only have to evaluate a series of simple • products. The final result is

$$(1,1) \otimes (1,1) = (0,0) \oplus (0,1) \oplus (0,2) \oplus (0,3) \oplus 2(2,0) \oplus 2(2,1) \oplus (2,2) \oplus (4,0). \tag{A3}$$

A2. Calculation of fusion rules for so(5) using formula (1.3)

The implementation of formula (1.3) is very simple. At first we fix the level and consider the fusion rule corresponding to a given tensor product; say (A3). For $\mathrm{sp}(4) = \mathrm{so}(5)$, the affine extension of $\bar{\lambda} = (\lambda_1, \lambda_2)$ is $\lambda = [k - \lambda_1 - \lambda_2, \lambda_1, \lambda_2]$. If on the right-hand side of (A3) there are terms with $k - \nu_1 - \nu_2 < 0$, one performs a shifted Weyl transformation of these weights in order to transform them into integrable weights (for which all three Dynkin indices are positive). It is the latter that appear in the final form of the product. They appear with a sign given by the sign of the Weyl transformation necessary to make them integrable. When a non-integrable weight cannot be Weyl reflected into an integrable one, it is ignored. There are three fundamental

Weyl reflections, s_0, s_1 and s_2, whose shifted actions on a weight $\lambda = [\lambda_0, \lambda_1, \lambda_2]$ are (here $\rho = [1,1,1]$):

$$s_0.\lambda = s_0(\lambda+\rho) - \rho = [-\lambda_0 - 2, 2\lambda_0 + \lambda_1 + 2, \lambda_2] \tag{A4a}$$

$$s_1.\lambda = [\lambda_0 + \lambda_1 + 1, -\lambda_1 - 2, \lambda_1 + \lambda_2 + 1] \tag{A4b}$$

$$s_2.\lambda = [\lambda_0, \lambda_1 + 2\lambda_2 + 2, -\lambda_2 - 2] \tag{A4c}$$

recall that $s_\mu \lambda = \lambda - \lambda_\mu \alpha_\mu$ (no sum), and here

$$\alpha_0 = [2,-2,0] \qquad \alpha_1 = [-1,2,-1] \qquad \alpha_2 = [0,-2,2].$$

In considering the affine extensions of weights $\bar{\nu}$ occuring in a given tensor product, only ν_0 can be negative. Therefore the last factor (the right-most term) of the Weyl element implementing the transformation into an integrable weight is necessarily s_0. If $\nu_0 = -1$, the shifted action of s_0 on the weight is neutral, so that the weight can be ignored. If $\nu_0 = -2$, (A4*a*) shows that it is sufficient to consider only s_0. For $\nu_0 < -2$, more complicated combinations of the simple Weyl reflections are required. The sign of an element of the Weyl group w is $(-)^l$ where l is the number of s_μ appearing in the decomposition of w. (For instance the sign of $w = s_\mu$ is -1.)

Let us now work out the fusion rules corresponding to the product (A3) level by level. Since the weights which are multiplied together are integrable only if $k \geqslant 2$, one first starts by considering the case where $k = 2$. The $k = 2$ affine extension of all the weights appearing on the right-hand side of (A3) is

$$[2,0,0] \qquad [1,0,1] \qquad [0,0,2] \qquad [-1,0,3] \qquad 2[0,2,0]$$
$$2[-1,2,1] \qquad [-2,2,2] \qquad [-2,4,0].$$

The fourth and the sixth ones are ignored since ν_0=-1. Furthermore one has

$$s_0.[-2,2,2] = [0,0,2] \qquad s_0.[-2,4,0] = [0,2,0].$$

and these two will appear with a minus sign. Thus one has

$$[0,1,1] \times [0,1,1] = [2,0,0] + [1,0,1] + [0,2,0].$$

At level $k = 3$, the affine extension of all the weights on the right-hand side of (A3) is

$$[3,0,0] \qquad [2,0,1] \qquad [1,0,2] \qquad [0,0,3] \qquad 2[1,2,0]$$
$$2[0,2,1] \qquad [-1,2,2] \qquad [-1,4,0].$$

These are all integrable except for the last two, for which ν_0=-1 and which are then ignored. Thus one finds

$$[1,1,1]\times[1,1,1] = [3,0,0]+[2,0,1]+[1,0,2]+[0,0,3]+2[1,2,0]+2[0,2,1].$$

For level k =4, the affine extensions of all the weights in (A3) are integrable so that there is no truncation of the tensor product coefficients. The final result for all levels can be written in a compact way, in the form (A3), by indicating with subscripts the minimum value of the level (that is k_0) at which each term appears. The result is

$$(1,1)\otimes(1,1)=(0,0)_2\oplus(0,1)_2\oplus(2,0)_2\oplus(2,0)_3\oplus(0,2)_3\oplus(0,3)_3\oplus 2(2,1)_3 \oplus(2,2)_4\oplus(4,0)_4. \tag{A5}$$

One sees that in the tensor product (A3), there are two factors of (2,0) but at level 2 one is cancelled by the shifted Weyl reflection of (4,0). The second factor (2,0) will then appear only at level 3. Similarly the term (0,2) is cancelled at level 2 and its first appearance is at level 3. This example illustrates very clearly the fact that the formula (1.2) is not a manifestly positive presentation of the fusion rules since the shifted Weyl action induces minus signs.

References

[1] Knizhnik V G and Zamolodchikov A B 1984 *Nucl. Phys.* B **247** 83
[2] Gepner D and Witten E 1986 *Nucl. Phys.* B **278** 493
[3] Forgacs P, Horvath Z, Palla L and Vecsernyes P 1988 *Nucl. Phys.* B **308** 477
[4] Gepner D Fusion rings and geometry *Preprint* (Santa Barbara) NSF-ITP-90-184
[5] Mathieu P, Sénéchal D and Walton M A 1991 work in progress
[6] Verlinde E 1988 *Nucl. Phys.* B **300** 389
[7] Weyl H 1946 *The Classical Groups* (Princeton: Princeton University Press)
[8] Walton M A 1990 *Nucl. Phys.* B **340** 777
[9] Spiegelglas M 1990 *Phys. Lett.* **245B** 169
[10] Walton M A 1990 *Phys. Lett.* **241B** 365
[11] Kac V G 1990 *Infinite Dimensional Lie Algebras* 3rd edn (Cambridge: Cambridge University Press) exercise 13.35
[12] Furlan P, Ganchev A and Petkova V B 1990 *Nucl. Phys.* B **343** 205
Fuchs J and van Driel P 1990 *Nucl. Phys.* B **346** 632
[13] Cummins C J, Mathieu P and Walton M A 1991 *Phys. Lett.* **254B** 390
[14] Bouwknegt P, McCarthy J and Pilch K *Preprint* MIT to appear in *Prog. Theor. Phys.*
[15] Cummins C J 1991 *J. Phys. A: Math. Gen.* **24** 391
[16] Lu S 1990 *PhD Thesis* MIT
[17] Patera J and Sharp R T 1979 *Lecture Notes in Physics* vol 94 (New York: Springer)
Couture M, Cummins C J and Sharp R T 1990 *J. Phys. A: Math. Gen.* **23** 1929
[18] Hongoh M, Sharp R T and Tilley D E 1974 *J. Math. Phys.* **15** 782
[19] Zamolodchikov A B and Fateev V A 1986 *Sov. J. Nucl. Phys.* **43** 657
[20] Black G R E, King R C and Wybourne B G 1983 *J. Phys. A: Math. Gen.* **16** 1555
Cummins C J 1990 Young diagrams and branching rules *Preprint* Concordia
[21] Kac V G and Peterson D 1984 *Adv. Math.* **53** 125